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Exact Solutions of a Fokker–Planck Equation

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Exact explicit solutions are given for a one-dimensional Fokker–Planck equation with a particular potential form involving hyperbolic functions. This potential contains four arbitrary parameters that can be chosen so that the potential is bistable. The solutions also contain parameters that can be chosen so that the initial distribution is approximately Gaussian, centered either at the unstable potential maximum or in the neighborhood of the secondary minimum. The use of the solutions to approximate solutions for other potentials is considered.

KEY WORDS: Bistable potentials; Fokker–Planck equation; exactly solved models.

1. INTRODUCTION

The Fokker–Planck equation has many applications,[1] but not many closed-form solutions are available. Apart from the similarity solutions,[2] which involve the hypergeometric function, most analytic work has been based on a transformation to a Schrödinger-type equation, followed by expansions in the eigenfunctions of this equation. In only a few examples can the resulting series solutions be summed to closed-form expressions.

In the last few years the number of known solutions of the Schrödinger equation has been increased[3–5] by the use of the Darboux transformation, which connects solutions of different second-order linear differential equations. The application of this work to the one-dimensional Fokker–Planck equation has given new solutions[6] for bistable potentials, in the form of eigenfunction expansions, the Darboux procedure generating the individual eigenfunctions. However, in cases where the eigenfunction expansion can be avoided[7] in the Schrödinger equation, it is also

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unnecessary in the corresponding Fokker–Planck equation, so that closed-form solutions are obtained. The purpose of this paper is to give various examples arising from the solvable Schrödinger potentials given by Sukumar.[5] The advantages of explicit solutions have been stressed recently[8] and some given for the half-range interval $[0, \infty)$. The solutions given in this paper are for the full interval $(-\infty, \infty)$.

New solutions of the Fokker–Planck equation can also be generated[9] using the Gelfand–Levitan method to change the associated Schrödinger equation. This has been shown[10] to be equivalent to using two Darboux transformations.

In the next section the relation between the one-dimensional Schrödinger and Fokker–Planck equations is summarized in a form convenient for later use and the application of the Darboux transformation is described. Then examples are given of exact, explicit solutions for a potential with parameters that can be chosen so that it is bistable. Section 4 discusses the choice of potential parameters to simulate a given potential of a different form.

2. FOKKER–PLANCK AND SCHRÖDINGER EQUATIONS

The Fokker–Planck equation[1] can be taken in the form

$$\frac{\partial W}{\partial t}=h^2\frac{\partial^2 W}{\partial x^2}+\frac{\partial}{\partial x}\left\{\frac{du}{dx}W\right\} \tag{2.1}$$

where $u(x)$ is the Fokker–Planck potential. Since $W(x, t)$ is a probability density, a solution of physical interest should be nonnegative and normalizable to satisfy

$$\int_{-\infty}^{\infty} W(x, t)\,dx=1 \tag{2.2}$$

The time-independent function

$$w(x)=e^{-u(x)/h^2} \tag{2.3}$$

satisfies (2.1), and, if normalizable, is the stationary solution, which is[1] the final form of any solution as $t\to\infty$.

The Schrödinger-type equation associated with (2.1) is

$$\frac{\partial\psi}{\partial t}=h^2\frac{\partial^2\psi}{\partial x^2}-V(x)\psi \tag{2.4}$$

where

$$V=\frac{1}{4h^2}\left(\frac{du}{dx}\right)^2-\frac{1}{2}\frac{d^2u}{dx^2} \tag{2.5}$$

If ψ satisfies (2.4), then $W=\psi\exp(-u/2h^2)$ satisfies (2.1). Also, $\exp(-u/2h^2)$ satisfies (2.4); if normalizable, it is the lowest eigenfunction of the operator $-h^2\,d^2/dx^2+V(x)$, belonging to the eigenvalue zero.

Any known solution of a time-dependent Schrödinger equation gives a solution of (2.4) on converting to imaginary time. A corresponding Fokker–Planck potential $u(x)$ may be found by equating $\exp(-u/2h^2)$ to the ground-state eigenfunction (if this exists). Although a constant can be added to the Schrödinger potential $V(x)$ so that zero is the ground-state eigenvalue, for the work in this paper it is more convenient to allow a non-zero lowest Schrödinger eigenvalue λ:

$$\left[-h^2\frac{d^2}{dx^2}+V(x)\right]e^{-u/2h^2}=\lambda e^{-u/2h^2} \tag{2.6}$$

This requires solutions of (2.1) and (2.4) to be related by

$$W(x,t)=\psi(x,t)\,e^{\lambda t}e^{-u/2h^2} \tag{2.7}$$

while in (2.5), V is increased by λ.

Actually, $\exp(-u/2h^2)$ in (2.7) can be any positive solution of (2.6), i.e., does not have to be normalizable, so that λ need not be an eigenvalue. For example, if $V(x)=0$ (the heat equation), a solution of (2.4) is

$$\psi(x,t)=(1+2\beta h^2t)^{-1/2}\exp[-\beta(x-a)^2/2(1+2\beta h^2t)] \tag{2.8}$$

and $\cosh\alpha x$ satisfies (2.6) with $\lambda=-h^2\alpha^2$. Then

$$W(x,t)=\psi(x,t)\cosh\alpha x\,e^{-h^2\alpha^2t} \tag{2.9}$$

is a solution of (2.1) with the "inverted" potential $u(x)=h^2\log(\operatorname{sech}^2\alpha x)$.

The Darboux transformation uses an operator[3-5]

$$B=-h\frac{\partial}{\partial x}+\frac{h}{\phi}\frac{d\phi}{dx} \tag{2.10}$$

where ϕ is a positive solution of

$$-h^2\frac{d^2\phi}{dx^2}+V(x)\phi=\mu\phi \tag{2.11}$$

Then[7] if $\psi(x, t)$ satisfies (2.4), $\tilde{\psi} = B\psi$ satisfies a similar equation with V replaced by

$$\tilde{V} = V - 2\frac{d}{dx}\left(\frac{1}{\phi}\frac{d\phi}{dx}\right) \tag{2.12}$$

and $\tilde{V}$ and $\tilde{\psi}$ can be used in (2.6) and (2.7) to obtain Fokker–Planck solutions.

The operator B also transforms solutions (in particular, eigenfunctions) of the time-independent equation into solutions for the new potential. Another solution of (2.11) when $\tilde{V}$ replaces V is $1/\phi$.

For the work below it will be sufficient to take $1/\phi$ for $\exp(-u/2h^2)$ in (2.6) and $\lambda = \mu$. Then, using $\tilde{\psi}$, (2.7) becomes

$$W_1(x, t) = -h\frac{e^{\mu t}}{\phi}\left(\frac{\partial\psi}{\partial x} - \frac{\psi}{\phi}\frac{d\phi}{dx}\right) = -he^{\mu t}\frac{\partial}{\partial x}\left(\frac{\psi}{\phi}\right) \tag{2.13}$$

The normalization integral (2.2) is therefore

$$\int_{-\infty}^{\infty} W_1(x, t)\, dx = -he^{\mu t}\left[\frac{\psi}{\phi}\right]_{-\infty}^{\infty}$$

In cases where the Fokker–Planck potential $u(x)$ satisfies $u(\pm\infty) \to \infty$, $1/\phi \to 0$, and $\int_{-\infty}^{\infty} W_1\, dx = 0$. However, $w = \exp(-u/h^2) = \phi^{-2}$ is then a normalizable stationary solution, and adding a multiple of W_1 will not change the normalization. Thus, solutions of (2.1) with potential $u = 2h^2 \ln \phi$ are given by[11]

$$W(x, t) = ke^{\mu t}\frac{\partial}{\partial x}\left(\frac{\psi}{\phi}\right) + \frac{N}{\phi^2} \tag{2.14}$$

where N^{-1} is the normalization constant $\int_{-\infty}^{\infty} \phi^{-2}(x)\, dx$. The constant k is arbitrary, but for a physical solution will be restricted by the condition $W(x, t) \geqslant 0$.

For example, take ψ from (2.8), and $\phi(x) = \cosh vx$ as the solution of (2.11), with $\mu = -h^2v^2$. Writing $\rho = (1 + 2\beta h^2 t)^{-1}$, one finds that (2.14) is given by

$$\begin{aligned} W(x, t) = &-k\rho^{1/2}(\operatorname{sech} vx)[v \tanh vx + \rho\beta(x - a)] \\ &\times \exp[-v^2h^2t - \beta\rho(x-a)^2/2] + \tfrac{1}{2}v \operatorname{sech}^2 vx \end{aligned} \tag{2.15}$$

which is a solution of the Fokker–Planck equation (2.1) with the potential

$$u(x) = 2h^2 \log \phi(x) = h^2 \log(\cosh^2 vx)$$

Note that μ, which is negative but otherwise can be chosen arbitrarily, is the Schrödinger eigenvalue.

The type of Darboux transformation considered here always gives a new Schrödinger potential with one extra prescribed eigenvalue. By applying a succession of transformations, one can add an arbitrary number of prescribed eigenvalues.[5]

3. SOLUTIONS FOR BISTABLE POTENTIALS

Starting from potential zero, two Darboux transformations are sufficient to give potentials with two minima. The first is essentially that giving (2.15), which can be generalized by replacing vx by $vx+\omega$. Then $\operatorname{sech}(vx+\omega)$ is a Schrödinger eigenfunction for the eigenvalue $-h^2v^2$. Another solution of the Schrödinger equation, generated using (2.10), is

$$\begin{aligned}\phi(x) &= \left[-h\frac{d}{dx}+hv\tanh(vx+\omega)\right][-\sinh(\gamma x+\theta)]\\ &= h\gamma\cosh(\gamma x+\theta)-hv\tanh(vx+\omega)\sinh(\gamma x+\theta)\end{aligned} \tag{3.1}$$

This solution is not normalizable, but is positive if $\gamma > v > 0$, and can therefore be used as the $\phi(x)$ in (2.10) for the second Darboux transformation. It satisfies (2.11) with $\mu = -h^2\gamma^2$.

Thus, applying B given by (2.10) and (3.1), we have that

$$\psi(x,t) = B\{\rho^{1/2}[v\tanh(vx+\omega)+\beta\rho(x-a)]\exp[-\beta\rho(x-a)^2/2]\} \tag{3.2}$$

satisfies a Schrödinger equation. The Schrödinger potential $\tilde{V}$ is given by (2.12) with $V(x) = -2v^2\operatorname{sech}^2(vx+\omega)$, but its explicit form is not required here. The ground-state Schrödinger eigenfunction is $1/\phi$ belonging to the eigenvalue $-h^2\gamma^2$, and there is also an eigenfunction $B[\operatorname{sech}(vx+\omega)]$ belonging to the (previous) eigenvalue $-h^2v^2 > -h^2\gamma^2$. Using a method given by Zheng,[12] we obtain the normalization constant for the ground state as $N = \frac{1}{2}h^2\gamma(\gamma^2-v^2)$.

Solutions can now be written down for the Fokker–Planck equation with potential $2h^2\log\phi$. The normalized steady-state solution is

$$g(x) = h^2\gamma(\gamma^2-v^2)/2\phi^2(x) \tag{3.3}$$

A transient solution corresponding to the next Schrödinger eigenfunction is

$$f(x,t) = \sinh(\gamma x+\theta)\{\exp[(v^2-\gamma^2)h^2t]\}/\phi^2(x)\cosh(vx+\omega) \tag{3.4}$$

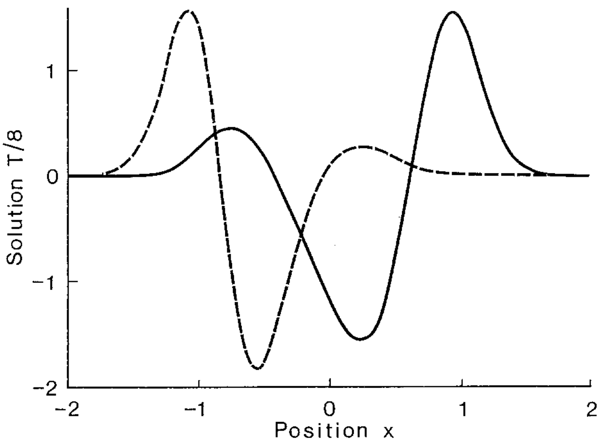
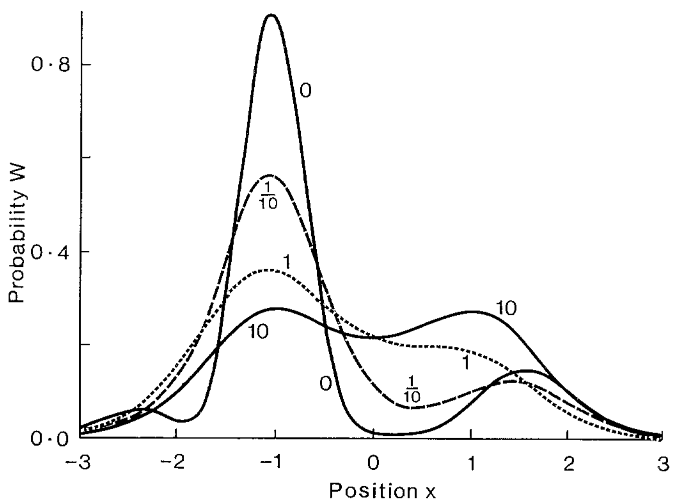


Fig. 1. Transient solutions for $h = 0.35355$, $\gamma = 2.67$, $v = 2.59$, $\theta = -0.63$, and $\omega = 0.285$. (—) $h^2T(6.4, 0.106; x, 0.0)$; (- -) $h^2T(8.0, -0.53; x, 0.0)$.



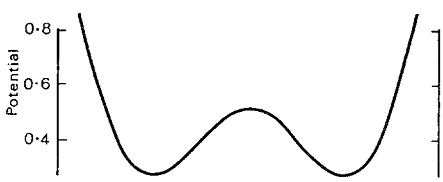


Fig. 2. Solution for $h = 1.0$, $\gamma = 1.293$, $v = 1.054$, and $\theta = \omega = 0$. The curves show $W(x, t)$ of (3.6) with $c = -0.177$, $\beta_1 = 4.7$, $a_1 = -1.1$, $k_1 = -0.095$, $\beta_2 = 1.0$, $a_2 = 0.5$, and $k_2 = 0.09$. Times shown are $t = 0$, (- -) 0.1, (···) 1.0, and 10.0 (the steady state shown in Fig. 5).

The transient solution $T(\beta, a; x, t)$ obtained from (3.2) is

$$\left\{-\beta\rho - v^2 + \beta^2\rho^2(x-a)^2 + \frac{(\gamma^2 - v^2)[\beta\rho(x-a) + v\tanh(vx+\omega)]}{\gamma\coth(\gamma x+\theta) - v\tanh(vx+\omega)}\right\}$$
$$\times \frac{\rho^{1/2}}{\phi(x)} \exp\left[-\gamma^2 h^2 t - \frac{\beta\rho(x-a)^2}{2}\right] \tag{3.5}$$

Since the solutions f and T have zero integral on $(-\infty, \infty)$, they must change sign, and are therefore not physically acceptable. For the potential with $v=2.59$, $\gamma=2.67$, $\omega=0.285$, and $\theta=-0.63$, two examples of the solutions $T(\beta, a; x, 0)$ are shown in Fig. 1.

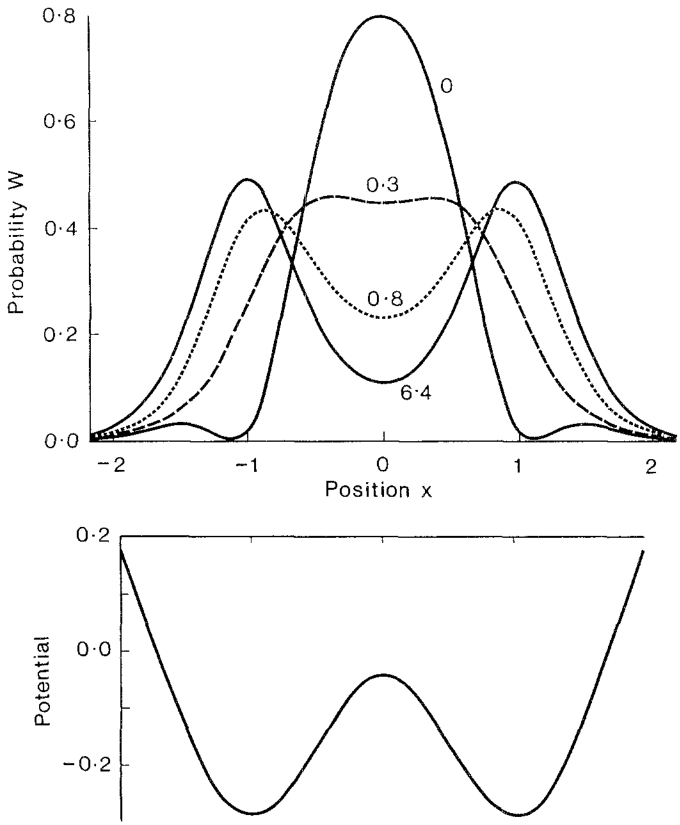


Fig. 3. Solution for $h=0.4082$, $\gamma=2.152$, $v=2.038$, and $\theta=\omega=0$. The curves show $W(x, t)$ of (3.6) with $c=0=a_1=a_2$, $\beta_1=3.6$, $k_1=-0.06833$, $\beta_2=0.9$, and $k_2=-0.014833$. Times shown are $t=0.0$, (- -) 0.3, (···) 0.8, and 6.4 (the steady state shown in Fig. 6).

Nonnegative solutions may be obtained by suitable superpositions of (3.3)–(3.5) of the general form

$$W(x, t) = g(x) + cf(x, t) + \sum_i k_i T(\beta_i, a_i; x, t) \tag{3.6}$$

Some examples are given in Figs. 2–4. Potentials with $\omega = \theta = 0$, as in Figs. 2 and 3, are even, and then ϕ and g will be even, f is odd, and T is even if $a = 0$.

The exponential factors in (3.4) and (3.5) make the solutions T decay faster than f; in Figs. 2 and 4 the solutions T are negligible for $t > 1.4$ and $t > 15$, respectively. The functions T contributing to the solutions in Fig. 4 are those shown in Fig. 1.

The solutions T also depend on t through the factor ρ, which causes the distribution to spread rather than move overall. Since the

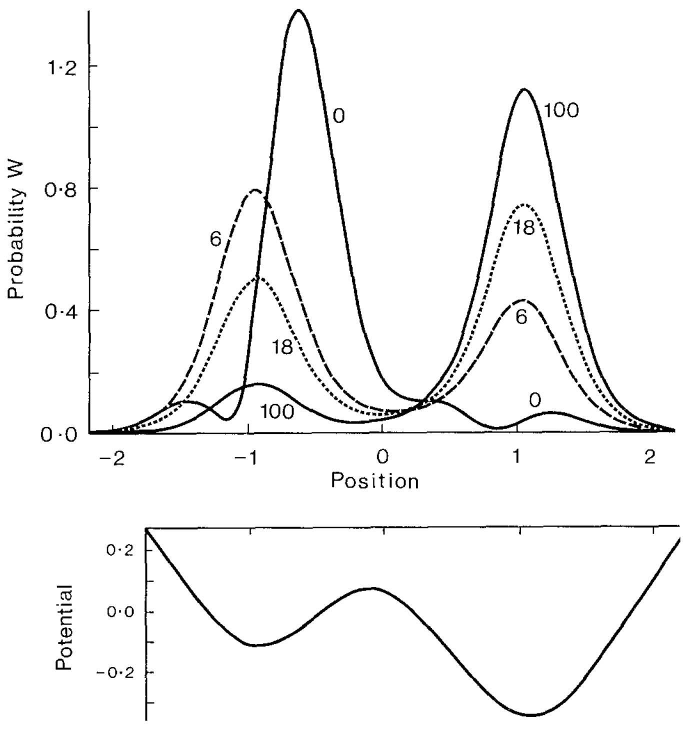


Fig. 4. Solution for $h = 0.35355$, $\gamma = 2.67$, $v = 2.59$, $\theta = -0.63$, and $\omega = 0.285$. The curves show $W(x, t)$ of (3.6) with $c = -0.13$, $\beta_1 = 6.4$, $a_1 = 0.106$, $k_1 = -0.01375$, $\beta_2 = 8.0$, $a_2 = -0.53$, and $k_2 = -0.0625$. Times shown are $t = 0.0$, 6.0, 18.0, and 100.0 (the steady state shown in Fig. 7).

Fokker–Planck equation (2.1) is invariant under time translation, one can replace t by $t-t_0$ in (3.5), which freedom may be useful in fitting a given initial state.

4. THE POTENTIAL

The potential $u=2h^2\log\phi$, with ϕ given by (3.1), contains four parameters v, γ, ω, and θ. Its asymptotic behavior is[2]

$$u(\pm\infty)\to 2h^2\{\log[\tfrac{1}{2}h(\gamma-v)]\pm\theta\pm\gamma x\} \tag{4.1}$$

What actually appears in the Fokker–Planck equation (2.1) is

$$u'=2h^2\phi'/\phi\to \pm 2h^2\gamma \qquad \text{as} \quad x\to\pm\infty \tag{4.2}$$

[2] Dimensional considerations indicate that it may be better to remove $\log[\frac{1}{2}h(\gamma-v)]$ (as an arbitrary constant change to the potential).

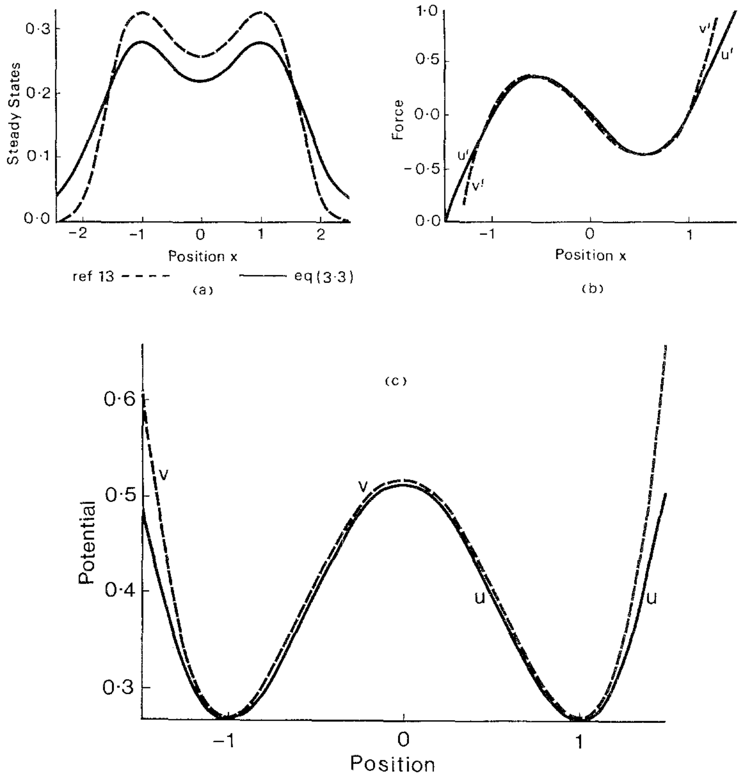


Fig. 5. Comparison of $v'(x)=x^3-x$ and $u'=2h^2\phi'/\phi$ with $h^2=1.0$, $v=1.054$, $\gamma=1.293$, and $\omega=\theta=0$. (a) Steady-state solutions, (b) v' and u', (c) v and u.

In applications the potential parameters have to be chosen to simulate the physical potential, especially in the region of the double minimum. Some aspects of this fitting procedure will now be discussed, using for a given physical potential v the model examples with[13–15] $v'(x) = x^3 - x$ and[16] $v'(x) = x^3 - x - 1/8$.

If $v'(x) = x^3 - x$, then

$$v(0) - v(1) = 1/4, \qquad v'(1) = v'(0) = 0, \qquad v''(1) = 2, \qquad v''(0) = -1 \tag{4.3}$$

are properties one would like to fit. For u to be even requires $\omega = \theta = 0$, giving $u'(0) = 0$, but leaving only two parameters γ and ν to be chosen to satisfy the remaining four conditions of (4.3). The best overall fit seems to be obtained by fitting the first two conditions. For $h = 1$ this gives $\nu = 1.054$, $\gamma = 1.293$, $u''(1) = 1.63$, $u''(0) = -1.1$; the overall result is illustrated in Figure 5. For $h = 0.4082$ ($h^2 = 1/6$), $\nu = 2.038$, $\gamma = 2.152$, $u''(1) = 1.36$, $u''(0) = -1.23$; the overall result is shown in Fig. 6. In each

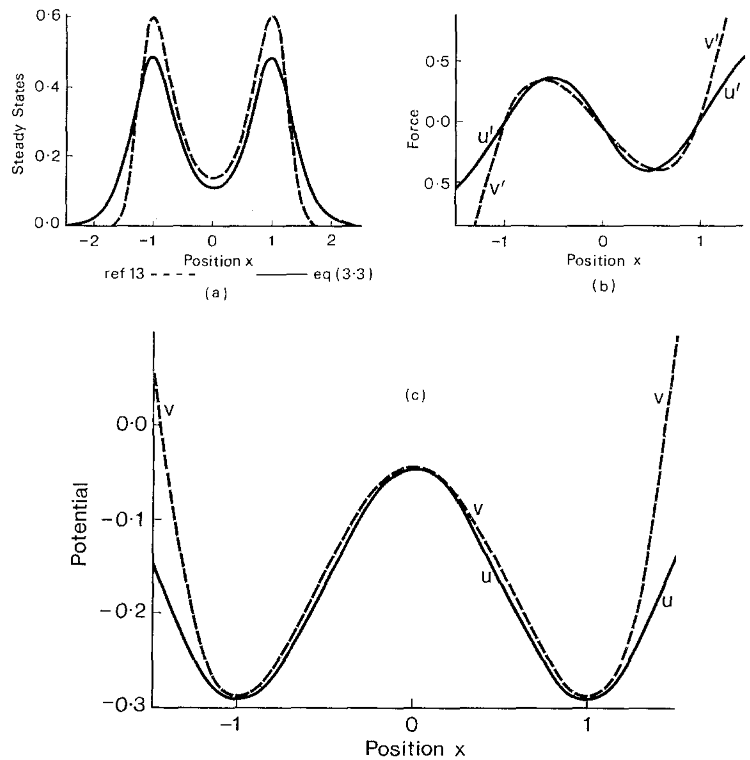


Fig. 6. Comparison of $v'(x) = x^3 - x$ and $u' = 2h^2\phi'/\phi$ with $h^2 = 1/6$. $\nu = 2.038$, $\gamma = 2.152$, and $\omega = \theta = 0$. (a) Steady-state solutions, (b) v' and u', (c) v and u.

figure a constant has been added to $v(x)$ so that $v(1)=u(1)$. The two potentials u are also shown in Figs. 2 and 3.

Least squares fits of the two parameters ν and γ to all conditions in (4.3) did not seem to give any improvement. Another possibility is to use a quasieven potential in which $u(x)$ is defined by (3.1) and $u=2h^2 \log \phi$ for $x \geqslant 0$, and then completed by $u(-x)=u(x)$. This allows ω and θ to be nonzero; fitting the last four conditions in (4.3) with the four parameters gives the results in Table I. The term "quasieven" indicates that higher odd derivatives of u may not be zero at $x=0$.

If $v'(x)=x^3-x-1/8$, then v' has zeros at $x= -0.9306$, -0.1271, and 1.058. The four parameters in u can be fixed by these three conditions on u' and also the value of the second derivative at one of the points. For example, for $h=0.35355$ $(h^2=1/8)$, requiring $u''(-0.9306)=v''(-0.9306)=1.597$ gives the parameters

$$\nu=2.59, \qquad \gamma=2.67, \qquad \omega=0.285, \qquad \theta= -0.63$$

and the results shown in Fig. 7. The potential is also shown in Fig. 4.

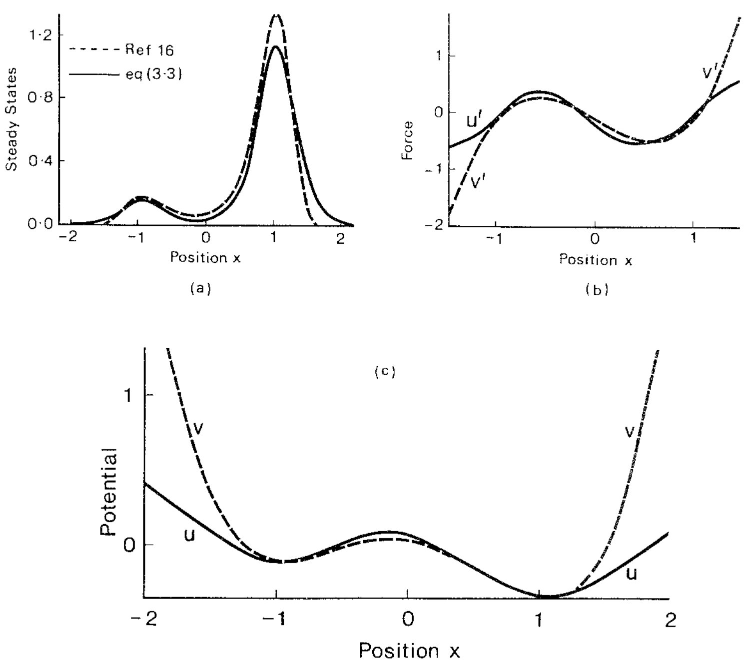


Fig. 7. Comparison of $v'(x)=x^3-x-1/8$ and $u'=2h^2\phi'/\phi$ with $h^2=1/8$, $\nu=2.59$, $\gamma=2.67$, $\omega=0.285$, and $\theta= -0.63$. (a) Steady-state solutions, (b) v' and u', (c) v and u.

Table I. Parameters for Quasieven Potentials Approximating $v' = x^3 - x$

h	h^2	ν	γ	ω	θ	$u(0)-u(1)$
1.	1.	1.14	1.44	−0.128	−0.274	0.27
0.4082	1/6	2.28	2.59	−0.261	−0.136	0.32
0.3162	0.1	2.88	3.22	−0.28	−1.74	0.31

By using further Darboux transformations, more potential parameters may be introduced. The required formulas for the corresponding Schrödinger equation have been given.[5] However, the three-parameter even potential (corresponding to three Schrödinger eigenvalues introduced by three Darboux transformations) was not useful, since there was always a minimum at $x = 0$.

5. DISCUSSION AND CONCLUSION

This paper has given exact, explicit solutions involving only elementary functions for a Fokker–Planck equation with a potential of the form $u = 2h^2 \log \phi$ with $\phi(x)$ given in (3.1). Suitable linear combinations of these solutions represent the decay of distributions corresponding to an initial concentration at either the unstable maximum point (Fig. 3) or the secondary minimum point (Figs. 2 and 4) of a bistable potential. In the potential there are four parameters that can be chosen to approximate any other required form, while the solutions contain parameters that can be chosen to simulate a required initial configuration.

The examples illustrated correspond to certain previous calculations. The potentials used in Figs. 2 and 3 were obtained by fitting $u'(x)$ to $x^3 - x$, and the distributions shown correspond to results exhibited by Baibuz *et al.*[13] in their Figs. 1 and 2. The potential used in Fig. 4 was obtained by fitting to $x^3 - x - 1/8$, and the distributions shown correspond to the results given in Fig. 5a of Tomita *et al.*[16]

The potential fitting, illustrated in Figs. 5–7, is poor outside the region between the two minima. Although this cannot be avoided using the potential obtained from (3.1), because u' and $v' = x^3 - x$ have different asymptotic forms, the defect is not too important when the initial distribution is mainly between the minima. Nevertheless, it will be worth extending the method to include potentials with different asymptotic behavior, which can be obtained using the Darboux transformation.[4,6]

The deviation of asymptotic forms also has a significant effect on the Schrödinger eigenvalues, which determine the decay rates of the transients appearing in an eigenfunction expansion. Since $u'(x) \to \pm 2h^2\gamma$ rather than $u'(x) \to \pm\infty$, the fit outside the minima is worse for smaller values of h. The potential obtained (see Table I) for $h^2 = 0.1$ has Schrödinger eigenvalues differing by $h^2(\gamma^2 - \nu^2) = 0.208$ compared to[14] 0.927 for $u'(x) = x^3 - x$.

Another matter requiring further work is the fitting of a given initial state to (3.6) with $t = 0$. For the initial states in Figs. 2–4 the values of c and k_i were obtained by requiring $W(x, 0)$ to be positive, but the β_i and a_i were chosen by trial and error. Some systematic method is desirable.

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